Nonparametric estimation of the local Hurst function of multifractional Gaussian processes

Jean-Marc Bardet

Nonparametric estimation of the local Hurst function of multifractional Gaussian processes

Joint paper with Donatas Surgailis (Lithuania)

Jean-Marc Bardet
bardet@univ-paris1.fr

SAMM, Université Paris 1

27 August 2012, Bucarest

Outline

1. Introduction

2. A new nonparametric estimator of the local Hurst function
   - First definition, assumptions and limit theorems
   - Second definition and limit theorems
   - Case of the General multifractional Brownian motion

3. Numerical comparison with the quadratic variations estimators
   - Definition and limit theorems
   - Numerical comparisons

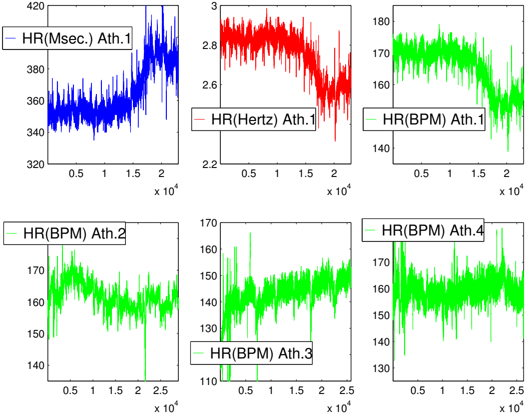
Outline

1. Introduction

2. A new nonparametric estimator of the local Hurst function
   - First definition, assumptions and limit theorems
   - Second definition and limit theorems
   - Case of the General multifractional Brownian motion

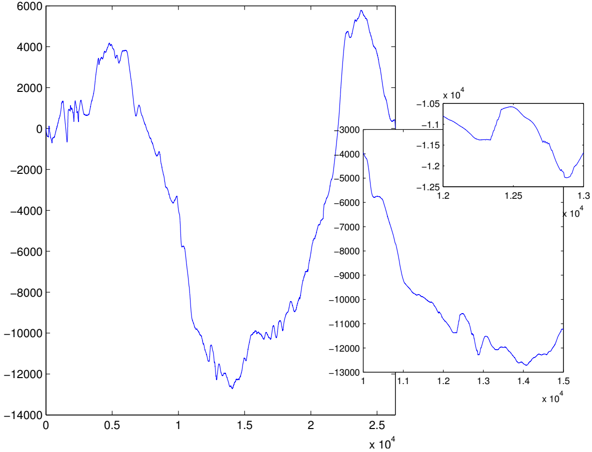
3. Numerical comparison with the quadratic variations estimators
   - Definition and limit theorems
   - Numerical comparisons

An athletic example...



Heart beat of athletes during Marathon de Paris 2004

A fact...



Successive zooms on the aggregated series of one athlete

The fractional Brownian motion

- $B^H = \{B^H_t, t \in \mathbb{R}\}$ fractional Brownian motion (FBM) with $H \in [0,1]$:

  B^H is a Gaussian centered process with stationary increments and $\mathrm{Var}\,(B^H_t) = \sigma^2 |t|^{2H}$.

Property

*A Gaussian process X having stationary increments H-***selfsimilar**

$\Longleftrightarrow$ *X F.B.M. with paremeter H*

- A trajectory of B^H is a.s. α-Hölderian for any $\alpha < H$:

  $\Longrightarrow$ H measures the local smoothness of B^H also called Hurst parameter

Two trajectories of FBM

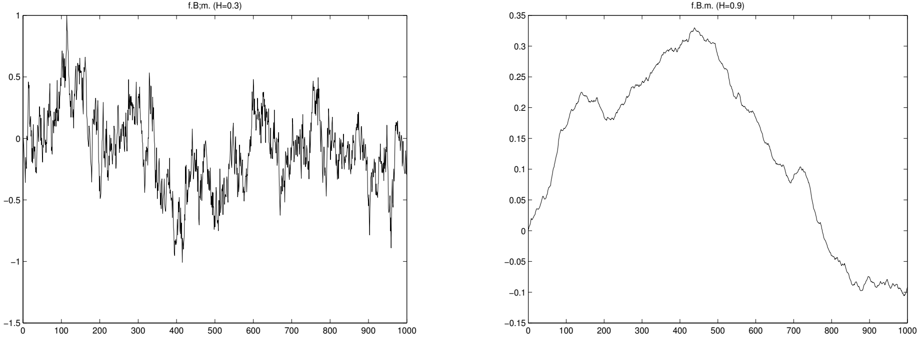


Figure: Trajectories of FBM with $H = 0.3$ (left) and $H = 0.9$ (right)

Other definitions of FBM

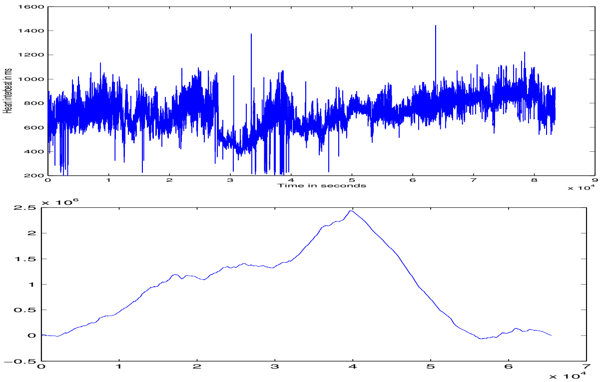
- Harmonizable representation:

$$B_t^H = C_1(H)\,\sigma^2 \int_{\mathbb{R}} \frac{e^{it\xi}-1}{|\xi|^{2H+1}}\,\widehat{W}(d\xi) \quad t \in \mathbb{R}$$

- Temporal representation:

$$B_t^H = C_2(H)\,\sigma^2 \int_{\mathbb{R}} \left((t-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right) W(du) \quad t \in \mathbb{R}$$

Another example



Heartbeats during 24h.

$\Longrightarrow$ H depending on t!

Multifractional Brownian motion

Two first versions:

- Harmonizable representation (Benassi *et al.*, 1997):

$$X_t = C_1(H(t))\,\sigma^2 \int_{\mathbb{R}} \frac{e^{it\xi}-1}{|\xi|^{2H(t)+1}}\,\widehat{W}(d\xi) \quad t \in \mathbb{R}$$

- Temporal representation (Peltier et Lévy-Véhel, 1995):

$$X_t = C_2(H(t))\,\sigma^2 \int_{\mathbb{R}} \big((t-u)_+^{H(t)-1/2} - (-u)_+^{H(t)-1/2}\big)\,W(du) \quad t \in \mathbb{R}$$

Aims

From an observed trajectory $\left(X_{\frac{1}{n}}, X_{\frac{2}{n}}, \ldots, X_1\right)$,

- Define a new non-parametric estimator $H(\cdot)$;
- Asymptotic properties of this estimator.
- Compare this estimator with the well-known quadratic variations estimator.

Outline

1. Introduction
2. A new nonparametric estimator of the local Hurst function
   - First definition, assumptions and limit theorems
   - Second definition and limit theorems
   - Case of the General multifractional Brownian motion
3. Numerical comparison with the quadratic variations estimators
   - Definition and limit theorems
   - Numerical comparisons

First definition

For $t_0 \in (0,1)$ and $\alpha \in (0,1)$, define:

$$\widehat{H}^{(IR)}_{n,\alpha}(t_0) := \Lambda^{-1}\Big(\frac{1}{2n^{1-\alpha}} \sum_{k=[nt_0-n^{1-\alpha}]}^{[nt_0+n^{1-\alpha}]} \frac{\big|\Delta_n^k X + \Delta_n^{k+1} X\big|}{\big|\Delta_n^k X\big| + \big|\Delta_n^{k+1} X\big|}\Big)$$

with

- $\Delta_n^k X = X_{\frac{k+2}{n}} - 2X_{\frac{k+1}{n}} + X_{\frac{k}{n}}$
- $\Lambda(h) = \mathbb{E}\Big[\frac{|\Delta_1^0 B^h + \Delta_1^1 B^h|}{|\Delta_1^0 B^h| + |\Delta_1^1 B^h|}\Big]$ for $h \in (0,1)$

$\Big(= \frac{1}{\pi}\arccos(-\rho_2(h)) + \frac{1}{\pi}\sqrt{\frac{1+\rho_2(h)}{1-\rho_2(h)}}\log\big(\frac{2}{1+\rho_2(h)}\big)$ with $\rho_2(h) = \frac{-3^{2h}+2^{2h+2}-7}{8-2^{2h+1}}\Big)$.

$\Longrightarrow$ Explanation...

Multifractional Gaussian processes?

Assumptions: $X=(X_t)_t$ is a centered Gaussian process such as:

(A)$_\kappa$ There exist η-Hölderian functions $0<H(t)<1$ and $c(t)>0$ for $t\in(0,1)$ such that for any $0<\varepsilon<1/2$ and $j\in\mathbb{Z}$,

$$\max_{[n\varepsilon]\leq k\leq[(1-\varepsilon)n]} n^\kappa \Big| \frac{\mathrm{Cov}\big(\Delta_n^k X\,,\,\Delta_n^{k+j}X\big)}{\mathrm{Cov}\big(\Delta_n^k B^{H(k/n)}\,,\,\Delta_n^{k+j}B^{H(k/n)}\big)} - c(\frac{k}{n})\Big| \underset{n\to\infty}{\longrightarrow} 0.$$

(B)$_\alpha$ There exist $C>0$, $\gamma>1/2$ and $0\leq\theta<\gamma/2$ such that for any $n\in\mathbb{N}^*$ and $1\leq k,k'<n-q$

$$\Big|\mathrm{Cor}\big(\Delta_n^k X,\Delta_n^{k'}X)\big)\Big| \;\leq\; C\,n^{(1-\alpha)\theta}\big(|k'-k|\wedge n^{1-\alpha}\big)^{-\gamma}.$$

Limit theorems for multifractional Gaussian processes

Theorem

Under Assumptions $\mathbf{(A)}_\kappa$ *and* $\mathbf{(B)}_\alpha$, *for all* $t_0 \in (0,1)$,

- *If* $0 < \alpha < \frac{\gamma-2\theta}{2(\gamma-\theta)}$,

$$\widehat{H}^{(IR)}_{n,\alpha}(t_0) \xrightarrow[n\to\infty]{a.s.} H(t_0).$$

- *If* $\kappa \geq \mu$ *and* $\frac{2\gamma-2\theta}{3\gamma-2\theta+4\gamma(\eta\wedge 2)} \leq \alpha < \frac{2\gamma-2\theta}{3\gamma-2\theta}$ *then for any* $\epsilon > 0$

$$\sup_{\epsilon<t<1-\epsilon} \left|\widehat{H}^{(IR)}_{n,\alpha}(t) - H(t)\right| = O_p(n^{-\mu}).$$

- *If* $\kappa \geq \mu_1$ *and* $\frac{\gamma-2\theta}{3\gamma-2\theta+4\gamma(\eta\wedge 2)} \leq \alpha < \frac{\gamma-2\theta}{3\gamma-2\theta}$ *then for any* $\epsilon > 0$, $\delta > 0$

$$\sup_{\epsilon<t<1-\epsilon} \left|\widehat{H}^{(IR)}_{n,\alpha}(t) - H(t)\right| = O(n^{-(\mu_1-\delta)}) \quad a.s.$$

Central Limit theorem for multifractional Gaussian processes

Theorem

Let $Z=(Z(t))_{t\in(0,1)}$ be a zero-mean Gaussian process satisfying $\mathbf{(A)}_\kappa$ *and* $\mathbf{(B)}_\alpha$*, with* $\alpha > \frac{1}{1+2(\eta\wedge 2)}$, $\kappa \geq \frac{1-\alpha}{2}$ *and* $\theta=0$. *Then for* $0<t_1<\cdots<t_u<1$,

$$\sqrt{2n^{1-\alpha}}\left(\widehat{H}^{(IR)}_{n,\alpha}(t_i)-H(t_i)\right)_{1\leq i\leq u} \quad \xrightarrow[n\to\infty]{\mathcal{D}} \quad \left(W^{(IR)}(t_i)\right)_{1\leq i\leq u},$$

where $W^{(IR)}(t_i)$, $i=1,\cdots,u$ *are inedependent centered Gaussian r.v.'s such as*

$$\mathbb{E}[W^{(IR)}(t_i)]^2 := \left[\frac{\partial}{\partial x}(\Lambda_2)^{-1}(\Lambda_2(H(t_i)))\right]^2\sigma^2(H(t_i))$$

where $$\sigma^2(H) := \sum_{k\in\mathbb{Z}} Cov\left(\psi\left(\Delta_1^0 B_H,\Delta_1^1 B_H\right),\psi\left(\Delta_1^k B_H,\Delta_1^{k+1} B_H\right)\right).$$

Proof: based on a CLT of triangular arrays of functional of Gaussian vectors

Theorem (Bardet et Surgailis, 2012)

Let $(\mathbf{Y}_n(k))_{1\leq k\leq n, n\in\mathbb{N}}$ *be a triangular array of standard Gaussian* $\mathbb{R}^\nu$*-vectors.*

- *For* $m \geq 1$, *there exists* $\rho : \mathbb{N} \to \mathbb{R}$ *such as for* $1 \leq p, q \leq \nu$,

$$\forall (j,k), \quad \left| \mathbb{E} Y_n^{(p)}(j) Y_n^{(q)}(k) \right| \leq |\rho(j-k)| \ \text{ with } \sum_{j\in\mathbb{Z}} |\rho(j)|^m < \infty;$$

- *For* $\tau \in [0,1]$ *and* $J \in \mathbb{N}^*$, *with* $(\mathbf{W}_\tau(j))_{j\in\mathbb{Z}}$ *a stationary Gaussian process*

$$(\mathbf{Y}_n([n\tau]+j))_{-J\leq j\leq J} \xrightarrow[n\to\infty]{\mathcal{D}} (\mathbf{W}_\tau(j))_{-J\leq j\leq J};$$

- *If* $\tilde{f}_{k,n} \in \mathbb{L}^2_0(\mathbf{X})$ $(n \geq 1, 1 \leq k \leq n)$ *with Hermite rank* $\geq m$ *and* $\exists \tilde{\phi}_\tau$, $\tau \in [0,1]$

such as $$\sup_{\tau\in[0,1]} \|\tilde{f}_{[\tau n],n} - \tilde{\phi}_\tau\|^2 = \sup_{\tau\in[0,1]} \mathbb{E}(\tilde{f}_{[\tau n],n}(\mathbf{X}) - \tilde{\phi}_\tau(\mathbf{X}))^2 \xrightarrow[n\to\infty]{} 0.$$

Theorem

Then with $\sigma^2 = \displaystyle\int_0^1 d\tau \Big(\sum_{j\in\mathbb{Z}} \mathbb{E}\big[\tilde{\phi}_\tau\left(\mathbf{W}_\tau(0)\right)\tilde{\phi}_\tau\left(\mathbf{W}_\tau(j)\right)\big]\Big) < \infty,$

$$n^{-1/2}\sum_{k=1}^{n}\tilde{f}_{k,n}\left(\mathbf{Y}_n(k)\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\big(0,\sigma^2\big).$$

Second definition

For t, $\alpha \in (0,1)$ and $j = 1, \ldots, p$,

$$\widehat{H}^{(IR)}_{n,\alpha,j}(t_0) := \Lambda_j^{-1}\Big(\frac{1}{2n^{1-\alpha}} \sum_{k=[nt_0-n^{1-\alpha}]}^{[nt_0+n^{1-\alpha}]} \frac{|\Delta_n^k X + \Delta_n^{k+j} X|}{|\Delta_n^k X| + |\Delta_n^{k+j} X|},\Big).$$

Theorem

Under Assumptions $\mathbf{(A)}_\kappa$ *and* $\mathbf{(B)}_\alpha$*, for all* $t_0 \in (0,1)$*,*

$$n^{(1-\alpha)/2}\big(\widehat{H}^{(IR)}_{n,\alpha,j}(t_0) - H(t_0)\big)_{1\leq j\leq p} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\Big(0\,,\,\Gamma(H(t_0))\Big)$$

Second definition (end)

Let $\widehat{\Gamma} := \Gamma(\widehat{H}^{(IR)}_{n,\alpha})$.

A new nonparametric estimator of $H(\cdot)$ using Pseudo-Generalized least squares:

$$\widetilde{H}^{(IR)}_{n,\alpha}(t_0) := \left(\mathbb{1}'_p \widehat{\Gamma}^{-1} \mathbb{1}_p\right)^{-1} \mathbb{1}'_p \widehat{\Gamma}^{-1} \left(\widehat{H}^{(IR)}_{n,\alpha,j}(t_0)\right)_{1\le j\le p}.$$

Theorem

Under Assumptions $\mathbf{(A)}_\kappa$ *and* $\mathbf{(B)}_\alpha$*, for all* $t_0 \in (0,1)$,

$$n^{(1-\alpha)/2}\left(\widetilde{H}^{(IR)}_{n,\alpha}(t_0) - H(t_0)\right) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\left(0\,,\,\left(\mathbb{1}'_p \Gamma^{-1}(H)\mathbb{1}_p\right)^{-1}\right)$$

Definition (Stoev and Taqqu, 2006)

On pose

$$Y_{(a^+,a^-)}(t) := K(H(t)) \int_{\mathbb{R}} \frac{e^{itx}-1}{|x|^{H(t)+\frac{1}{2}}}\, U_{(a^+,a^-)}(H(t),x)\, \widehat{W}(dx),$$

with for $h \in (0,1)$

$$\begin{aligned} U_{(a^+,a^-)}(h,x) \quad &:= \quad \frac{\left(a^+ e^{-i\,\mathrm{sign}(x)(h+\frac{1}{2})\frac{\pi}{2}} + a^- e^{i\,\mathrm{sign}(x)(h+\frac{1}{2})\frac{\pi}{2}}\right)}{\left((a^+)^2+(a^-)^2-2a^+a^-\sin(\pi\, h)\right)^{1/2}}. \\ &:= \quad \frac{1}{\sqrt{\pi}} \qquad \text{if } h=1/2 \text{ and } a^+=a^-. \end{aligned}$$

Estimation of $H(\cdot)$

Theorem

For a trajectory of the process $(Y_{(a^+,a^-)}(t))_t$,

- *If* $\max(0, 1-4((\eta \wedge 2) - H(t))) < \alpha < \frac{1}{2}$,

$$\widetilde{H}^{(IR)}_{n,\alpha}(t_0) \xrightarrow[n\to\infty]{a.s.} H(t_0)$$

- *If* $\max\left\{\dfrac{1}{1+2(\eta\wedge 2)}, 1-4\big(\eta \wedge 2 - H(t_0)\big)\right\} < \alpha < 1$,

$$n^{(1-\alpha)/2}\big(\widetilde{H}^{(IR)}_{n,\alpha}(t_0) - H(t_0)\big) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}\Big(0\,,\,\big(\mathbb{1}_p'\Gamma^{-1}(H)\mathbb{1}_p\big)^{-1}\Big)$$

Outline

1. Introduction

2. A new nonparametric estimator of the local Hurst function
   - First definition, assumptions and limit theorems
   - Second definition and limit theorems
   - Case of the General multifractional Brownian motion

3. Numerical comparison with the quadratic variations estimators
   - Definition and limit theorems
   - Numerical comparisons

Generalized quadratic variations estimator of $H(\cdot)$

Define (see Istas and Lang, 1994, Benassi *et al.*, 1998, Coeurjolly, 2005):

$$\widehat{H}_{n,\alpha}^{(QV)}(t_0) := \frac{1}{2}\,\frac{A'}{A'A}\Big(\log\Big(\frac{1}{2n^{1-\alpha}}\sum_{k=[nt_0-n^{1-\alpha}]}^{[nt_0+n^{1-\alpha}]}\big|X_{\frac{k+2j}{n}}-2X_{\frac{k+j}{n}}+X_{\frac{k}{n}}\big|^2\Big)\Big)'_{1\le j\le p}$$

with $A := \big(\log i - \frac{1}{p}\sum_{j=1}^p \log j\big)_{1\le i\le p} \in \mathbb{R}^p$.

Theorem

Mutatis mutandis, the limit theorems obtained for $\widetilde{H}_{n,\alpha}^{(IR)}(t_0)$ *also hold for* $\widehat{H}_{n,\alpha}^{(QV)}(t_0)$.

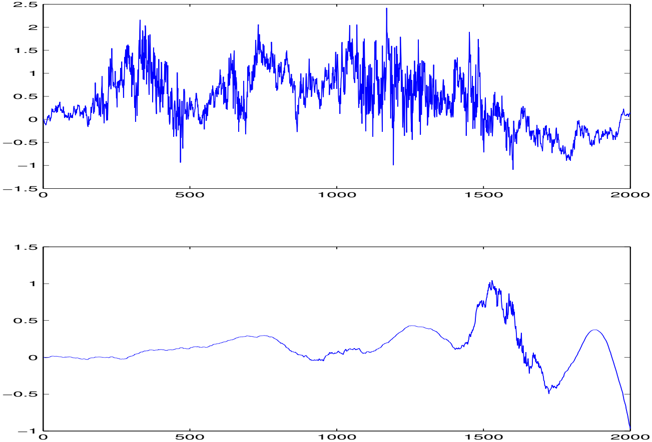


Figure: *Examples of MBM trajectories (up,* $H \in \mathcal{C}^{\eta-}$ *with* $\eta = 0.6$*, down* $H(t) = 0.1 + 0.8(1-t)\sin^2(10t)$*)*

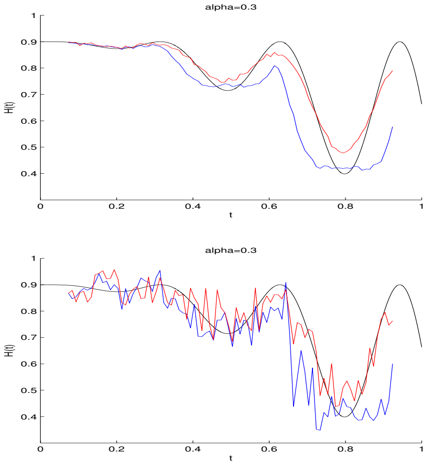


Figure: *Estimation of* $H_4(t) = 0.1 + 0.8(1-t)\sin^2(10t)$ *for* $n = 6000$, $\alpha = 0.3$ *(left) and* 0.4 *(right)*

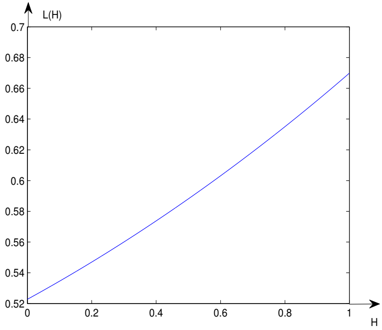
$H_1(t) = 0.6$	α	0.2	0.3	0.4	0.5
$n = 2000$	$\sqrt{\widehat{MSE}}$ for $\widehat{H}^{(QV)}_{n,\alpha}$	0.041	0.051	0.069	0.096
	$\sqrt{\widehat{MSE}}$ for $\widehat{H}^{(IR)}_{n,\alpha}$	0.061	0.077	0.106	0.145
$n = 6000$	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(QV)}_{n,\alpha}$	0.025	0.033	0.050	0.074
	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(IR)}_{n,\alpha}$	0.037	0.049	0.076	0.115
$H_2(t) = 0.1 + 0.8t$	α	0.2	0.3	0.4	0.5
$n = 2000$	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(QV)}_{n,\alpha}$	0.170	0.073	0.072	0.096
	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(IR)}_{n,\alpha}$	0.059	0.071	0.098	0.135
$n = 6000$	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(QV)}_{n,\alpha}$	0.114	0.044	0.048	0.070
	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(IR)}_{n,\alpha}$	0.036	0.046	0.069	0.103
$H_3(t) = 0.5 + 0.4\sin(5t)$	α	0.2	0.3	0.4	0.5
$n = 2000$	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(QV)}_{n,\alpha}$	0.362	0.123	0.080	0.096
	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(IR)}_{n,\alpha}$	0.093	0.071	0.091	0.124
$n = 6000$	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(QV)}_{n,\alpha}$	0.260	0.077	0.052	0.072
	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(IR)}_{n,\alpha}$	0.057	0.047	0.065	0.097
$H_4(t) = 0.1 + 0.8(1-t)\sin^2(10t)$	α	0.2	0.3	0.4	0.5
$n = 2000$	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(QV)}_{n,\alpha}$	0.320	0.164	0.117	0.112
	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(IR)}_{n,\alpha}$	0.165	0.098	0.091	0.112
$n = 6000$	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(QV)}_{n,\alpha}$	0.251	0.135	0.071	0.078
	$\sqrt{\widehat{MSE}}$ for $\widetilde{H}^{(IR)}_{n,\alpha}$	0.148	0.062	0.067	0.091

Table: Estimators $\widetilde{H}^{(QV)}_{n,\alpha}$ et $\widetilde{H}^{(IR)}_{n,\alpha}$ when $H(\cdot)$ is a $\mathcal{C}^\infty$ function.

$H \in \mathcal{C}^{1.5}(0,1)$	α	0.2	0.3	0.4	0.5
$n = 2000$	$\sqrt{\widehat{MISE}}$ for $\widetilde{H}^{(QV)}_{n,\alpha}$	0.261	0.112	0.085	0.098
	$\sqrt{\widehat{MISE}}$ for $\widetilde{H}^{(IR)}_{n,\alpha}$	0.098	0.077	0.093	0.128
$n = 6000$	$\sqrt{\widehat{MISE}}$ for $\widetilde{H}^{(QV)}_{n,\alpha}$	0.164	0.066	0.053	0.070
	$\sqrt{\widehat{MISE}}$ for $\widetilde{H}^{(IR)}_{n,\alpha}$	0.054	0.047	0.066	0.098
$H \in \mathcal{C}^{0.6}(0,1)$	α	0.2	0.3	0.4	0.5
$n = 2000$	$\sqrt{\widehat{MISE}}$ for $\widetilde{H}^{(QV)}_{n,\alpha}$	0.140	0.086	0.081	0.094
	$\sqrt{\widehat{MISE}}$ for $\widetilde{H}^{(IR)}_{n,\alpha}$	0.088	0.078	0.096	0.135
$n = 6000$	$\sqrt{\widehat{MISE}}$ for $\widetilde{H}^{(QV)}_{n,\alpha}$	0.130	0.067	0.056	0.071
	$\sqrt{\widehat{MISE}}$ for $\widetilde{H}^{(IR)}_{n,\alpha}$	0.066	0.052	0.067	0.103

Table: Estimators $\widetilde{H}^{(QV)}_{n,\alpha}$ et $\widetilde{H}^{(IR)}_{n,\alpha}$ when $H(\cdot)$ is a $\mathcal{C}^{\eta-}$ function.

Figures



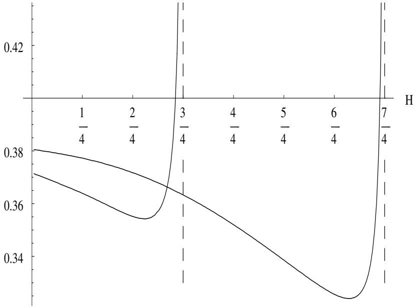


Figure: Graphs of functions $H \mapsto \Lambda(H)$ and $H \mapsto \sigma(H)$